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Internal Symmetry and Lorentz Invariance*

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The notion of a rigorous internal symmetry implies an over-all symmetry group G that contains the inhomogeneous Lorentz group as a proper subgroup. Such a rigorous symmetry does not automatically require degenerate mass multiplets. But over-all symmetry groups that are compatible with mass splittings are severely restricted as follows. Assume the generators of G are the Lorentz generators and the generators of either a semisimple or a compact Lie group. If the Cartan subalgebra of its semisimple part is Lorentz invariant, then all the generators of the internal symmetry are Lorentz invariant and therefore there can be no mass splitting. In particular, if the internal symmetry is SU(3) and T_z and Y are Lorentz invariant, then all the generators of SU(3) are Lorentz invariant.

THE known mesons, baryons, and their resonances have been successfully identified with irreducible representations of the group SU(3). The members of an SU(3) multiplet all have the same mass. This degeneracy can be resolved by introducing a mass term that is not invariant under SU(3). Lorentz invariance is treated completely separately from the internal symmetry.

An alternative approach that has been considered[1] is to search for an over-all symmetry group G that contains SU(3) and the inhomogeneous Lorentz group $\mathfrak{L}$ as subgroups but is not a direct product, so that at least one of the generators of SU(3) does not commute with all the Lorentz generators. Then an irreducible representation of G will already contain different masses. The usual classification can be retained only by requiring that the z component T_z of the isotopic spin, the magnitude $\mathbf{T}^2$, and the hypercharge Y commute with all the Lorentz generators.

We shall show that there are severe restrictions on such combined groups.[2] In particular, if the generators of the group G are assumed to consist only of the generators of SU(3) and $\mathfrak{L}$, and if T_z and Y commute with all the generators of $\mathfrak{L}$, then

$$G=\mathrm{SU}(3)\otimes\mathfrak{L}. \tag{1}$$

We shall prove a more general result. Let the internal symmetry group be assumed to be a semisimple group[3] of rank l. The canonical set of generators H_i and E_α satisfy the commutation relations

$$[H_i,H_j]=0 \quad (i,j=1,\cdots,l), \tag{2}$$

$$[H_i,E_\alpha]=R_i(\alpha)E_\alpha. \tag{3}$$

The H_i, provide the commuting observables that label states. [For SU(3), $l=2$, so there are two quantum numbers that are linear in the generators of the symmetry group.] Let the generators of the inhomogeneous Lorentz group be $L_\rho(\rho=1,\cdots,10)$ and

$$[L_\rho,L_\sigma]=\sum_\tau\lambda_{\rho\sigma}{}^\tau L_\tau, \tag{4}$$

where $\lambda_{\rho\sigma}{}^\tau$ are the structure constants of the Lorentz group. Let us assume that the quantum numbers are Lorentz invariant, i.e.,

$$[H_i,L_\rho]=0. \tag{5}$$

We then prove that

$$[E_\alpha,L_\rho]=0, \tag{6}$$

so that Eq. (1) holds.

Proof: The most general expression for $[E_\alpha, L_\rho]$ is

$$[E_\alpha,L_\rho]=\sum_\beta x_{\alpha\rho}{}^\beta E_\beta+\sum_j y_{\alpha\rho}{}^j H_j+\sum_\tau a_{\alpha\rho}{}^\tau L_\tau. \tag{7}$$

Consider the Jacobi identity

$$[[E_\alpha,L_\rho],H_i]+[[L_\rho,H_i]E_\alpha]+[[H_i,E_\alpha]L_\rho]=0. \tag{8}$$

From the commutation relations (2), (3), (5), and (7), we have

$$\sum_\beta(x_{\alpha\rho}{}^\beta R_i(\beta)-x_{\alpha\rho}{}^\beta R_i(\alpha))E_\beta -\sum_j y_{x\rho}{}^j R_i(\alpha)H_j-\sum_\tau a_{\alpha\rho}{}^\tau R_i(\alpha)L_\tau=0. \tag{9}$$

For every α, $R_i(\alpha)\neq 0$ for at least one value of i, and $R_i(\alpha)\neq R_i(\beta)$ for at least one value of i if $\alpha\neq\beta$. Therefore

$$y_{\alpha\rho}{}^i=0, \quad a_{\alpha\rho}{}^\tau=0, \tag{10}$$

and

$$x_{\alpha\rho}{}^\beta=\delta_\alpha{}^\beta x_{\alpha\rho}. \tag{11}$$

Next consider the Jacobi identity

$$[[L_\rho,L_\sigma],E_\alpha]+[[L_\sigma,E_\alpha],L_\rho]+[[E_\alpha,L_\rho],L_\sigma]=0. \tag{12}$$

Evaluation of the terms yields

$$-\sum_\tau\lambda_{\rho\sigma}{}^\tau x_{\alpha\tau}E_\alpha-x_{\alpha\sigma}x_{\alpha\rho}E_\alpha+x_{\alpha\rho}x_{\alpha\sigma}E_\alpha =-\sum_\tau\lambda_{\rho\sigma}{}^\tau x_{\alpha\tau}E_\alpha=0. \tag{13}$$

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[1] A. Barut, Coral Gables Conference on Symmetry Principles, January 1964 (unpublished); B. Kursunoglu, *ibid.*

[2] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964), first considered the problem of the compatibility of internal symmetry and Lorentz invariance.

[3] For background information about Lie algebras, cf. A. Salam, *Trieste Lectures on Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963).

Since, for every τ, we can choose ρ and σ so that $\lambda_{\rho\sigma}{}^{\tau}\neq 0$ and $\lambda_{\rho\sigma}{}^{\tau'}=0$ for $\tau'\neq\tau$, we conclude from (13) that $x_{\alpha\tau}=0$. Thus $[E_\alpha,L_\rho]=0$ and Eq. (1) holds.

The theorem also holds for any compact internal symmetry group.[4] Then, if the group is not semisimple, it is the direct product of a semisimple group and an Abelian group (toroid). The generators of the toroid commute with all the generators of the internal symmetry group, and the proof of Eq. (6) remains unchanged.

Our proof does not exclude the possibility that the internal symmetry group and the Lorentz group are embedded in some larger symmetry group.[5] If this is the case, one must face the problem of interpreting the additional symmetry operations associated with this larger group.

[4] L. Pontryagin, *Topological Groups* (Princeton University Press, Princeton, New Jersey, 1939), p. 282.

[5] For a particular attempt, cf. Ref. 1.

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Consequences of Crossing Symmetry in SU_3†

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The following processes of 2 octets transforming into 2 octets are discussed on the basis of crossing symmetry; $P+B\to P+B$, $B+B\to B+B$, $P+V\to P+V$, $\cdots$ and their crossed channels, where B, P, and V represent baryons, pseudoscalar mesons, and vector mesons, respectively. The relation between channel amplitudes, the number of independent channel amplitudes, and spin selection rules are systematically obtained.

A SET of states that transform into one another under the unitary transformations SU_3 will form multiplets that are labeled by two quantum numbers (λ,μ). In the octet model,[1,2] the baryons $B=(N,\Sigma,\Lambda,\Xi)$, the antibaryons $\bar{B}$, the pseudoscalar mesons $P=(K,\pi,\eta,\bar{K})$, and the vector mesons $V=(K^*,\rho,\phi^0,\bar{K}^*)$ are assigned to the (1,1) representation of the group SU_3. The amplitude $(ab|cd)$ for the reaction $a+b\to c+d$ can be described as two octets transforming into two other octets.

Two octets (1,1) can couple together to form the product representations (2,2), $(1,1)_s$, (0,0), (0,3), (3,0), and $(1,1)_a$. The representation $(1,1)_s$ transforms with a positive phase whereas the representation $(1,1)_a$ transforms with a negative phase under an R transformation[1] that is independent of SU_3. There are thus six channel amplitudes A_{27}, A_{8s}, A_1, $A_{\overline{10}}$, A_{10}, and A_{8a} which are diagonal elements of the S matrix for the representations (2,2), $(1,1)_s$, (0,0), (0,3), (3,0), and $(1,1)_a$, respectively. There are also two nondiagonal channel amplitudes A_{as} and A_{sa} that couple the representations $(1,1)_s$ and $(1,1)_a$.

One can sometimes obtain relations among the channel amplitudes by use of invariance under time reversal, $(ab|cd)\to(cd|ab)$, charge conjugation, $(ab|cd)\to(\bar{a}\bar{b}|\bar{c}\bar{d})$, and parity operation—all of which hold in the strong interactions—together with crossing symmetry. We shall see that most of the relations follow only from time-reversal invariance in the direct channel (channel I).

Let us define the three channels I, II, and III as follows:

$$\text{Channel I:}\quad (ab|cd),\quad \text{amplitude}=A;$$
$$\text{Channel II:}\quad (\bar{c}a|d\bar{b}),\quad \text{amplitude}=B;$$
$$\text{Channel III:}\quad (\bar{c}b|\bar{a}d),\quad \text{amplitude}=C.$$

Then the amplitudes A, B, and C are related to each other by crossing symmetry; i.e., $A=O_2B$ and $A=O_3C$, where the crossing matrices O_2 and O_3 are[3,4]

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[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); California Institute of Technology Report CTSL-20, 1961 (unpublished).

[2] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[3] From the crossing matrix for $A=O_2B$, one can obtain $A=O_3C$ in the following way:

$$(ab|cd)\to(ba|cd)\overset{O_3}{\to}(\bar{c}a|\bar{b}d)\to(\bar{c}a|d\bar{b}),$$

$$(ab|cd)\overset{O_2}{\to}(\bar{c}a|d\bar{b}).$$

This results in $A_a\to -A_a$, $A_{as}\to -A_{as}$, $C_a\to -C_a$, and $C_{sa}\to -C_{sa}$, where A_a represents $A_{\overline{10}}$, A_{10}, and A_{8a}.

[4] The crossing matrix has been considered by R. E. Cutkosky, Ann. Phys. **23**, 405 (1963); D. E. Neville, Phys. Rev. **132**, 844 (1963); J. J. de Swart, Nuovo Cimento **31**, 420 (1964). We thank S. Okubo and B. Lee for pointing this out. Equation (1) is read as

$$A_{27}=\frac{7}{40}B_{27}+\frac{1}{5}B_{8s}+\frac{1}{8}B_1-\frac{1}{12}B_{\overline{10}}-\frac{1}{12}B_{10}-\frac{1}{3}B_{8a},\ \text{etc.}$$

The implications of time-reversal in elastic scattering in connection with SU_3 have been remarked on by P. G. O. Freund, H. Ruegg, D. Speiser, and A. Morales, Nuovo Cimento **25**, 307 (1962); P. Tarjanne, Ann. Acad. Sci. Fennicae Ser. A VI, 105 (1962).